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Robust Stabilization of Uncertain Aircraft Active Systems

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Abstract: Often, dynamical aircraft systems cannot always be accurate because some approximation assumptions, imprecisions or uncertainties may have been introduced or imposed during the modeling process. Mathematical models of aircraft systems always contain uncertain elements, which model the designer's lack of knowledge about some parameter values, disturbances and unmodeled dynamics. Using both Lyapunov's direct method and the linear matrix inequality approach, we develop the controller design procedure, and give a definite feel for stability analysis and robust control for aircraft systems with significant uncertainty. An example of an uncertain thrust vectoring aircraft is studied to illustrate our control design strategy.

Key Words: Aircraft, uncertain systems, stabilization, linear matrix inequality

NOMENCLATURE

$\|\mathbf{R}\|$	the set of real numbers
$\|.\|$	the habitual Euclidean norm
I	the matrix identity with appropriate dimension
A'	the matrix transpose of A
$\lambda_{\min}(A)$	the smallest eigenvalue of the matrix A
$\lambda_{\max}(A)$	the largest eigenvalue of the matrix A
$A > B$	the matrix A–B is positive definite
$\mu(A)$	the measure of the matrix A equal to $\lambda_{\max}(A + A')/2$
C_n^k	the binomial coefficient

1. INTRODUCTION

Most aircraft systems encountered in practice have highly uncertain dynamics, and their fundamental structure is very complex. However, the emphasis in robust control, especially from the point of view of assessing performance, has been on cases where uncertainty is either small or has a known parametrization. This gap between existing methods and the reality of many practical plants has created a clear need for a general approach to the design of feedback schemes for situations where the aircraft plant uncertainty is large and its structure not known a priori.

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Several papers have dealt with classes of linear uncertain dynamical systems, which are affine in the control input. To encompass all possible realizations of uncertainty in the systems, differential inclusion is adopted to model this class of uncertain dynamic systems. Deterministic feedback control proposes the use of linear or nonlinear feedback control functions, which operate effectively over a specified magnitude range of system parameter variations and disturbances. Applying these deterministic methods, the synthesis of stabilizing feedback controls may give rise to controls with "high" gain in order to address the problem of uncertainty in the system. For all physically realistic systems, however, there is some implicit bound on the gain that is allowable for such stabilizing controllers. Therefore, it is necessary to consider the problem of constructing a stabilizing controller, whose gain satisfies a constraint defined in terms of a practically acceptable threshold level.

Robust aircraft control plays a greater role in achieving high performance for aircraft systems that are too complex to be modeled accurately. We refer the reader to Kogan (1999), Najson and Kreindler (1996), Ni and Chen (1996), Xie and Soh (1994), and Zhoo and Khargonekar (1988) to see some advances in this domain. Linear matrix inequalities (LMIs) have recently emerged as powerful tools for stabilization of linear uncertain systems (Boyd et al., 1994), where the uncertainties can appear in the state matrix, the control matrix, or in both the state and control matrices. This technique is very attractive and is widely applicable, not only in optimization problems but also in many areas where uncertainty arises. With the development of numerical methods as interior point techniques for solving LMI problems, the LMI approach has gained increased interest and it has been adopted for numerous engineering problems.

In this paper we investigate some aspects of the linear robust control of imperfectly known (uncertain) aircraft systems, such as active control systems, which are studied in the field of aeroservoelasticity. The main emphasis is on the deterministic design of a linear feedback control for aircraft dynamical systems in order to achieve some desired performance, via an approach which does not require full identification of the uncertain elements. First, under certain conditions, we propose an LMI-based robust controller to stabilize uncertain plants having uncertainties in the state matrix. Then, a new design methodology will be addressed to stabilize linear systems with lower-triangular uncertainties. Under certain assumptions on the nominal matrices, a Lyapunov-like equation is proposed to stabilize uncertain plants, having imperfectly known time-varying parameters in both the state and control matrices. We show that the conditions of existence of a stabilizing controller are not severe by comparison with some matching conditions proposed in the literature. Our conditions are not coupled and only express the boundedness of the uncertain parts.

Furthermore, the stabilization of the uncertain plant turns on a simple calculation of the solution of Lyapunov matrix equation. Finally, an example of uncertain thrust vectoring aircraft is given to illustrate the efficiency of the control design proposed here.

2. PROBLEM STATEMENT

In aircraft modeling, the determination of a mathematical model of any aircraft nearly always involves approximations and, hence, imprecision or uncertainty is introduced during the modeling procedure. Furthermore, existing processes are frequently subject to extraneous disturbances and, in addition, there may be imperfectly known inputs.

The goal of this section is the determination of a simple design methodology to control uncertain aircraft systems with robust linear controllers. This constitutes the overall motivation for this research. Particular questions that we aim to discuss here, and which are the subject of ongoing research, are summarized as follows. (i) How does one develop feedback controller to best overcome the effects of uncertainties presenting in high-dimensional aircraft? (ii) What are the key properties of a set of plant uncertainties which limit the ability of feedback to improve performance and how should the controller be designed to be insensitive to the different structures of uncertainties?

Definition 1. *The system (2) is said to be quadratically stabilizable if there exists a continuous $v(.) : \mathbf{R}^n \mapsto \mathbf{R}^m$, with $v(0) = 0$ an $n \times n$ positive definite matrix P and a constant $\beta > 0$ such that for any admissible uncertainty $\Delta A(t) \in \mathbb{F} \subset \mathrm{R}^{n\times n}$, for the Lyapunov function $V(x) = x'Px$, the derivative $\dot{V}$, corresponding to the closed loop-system with the feedback law $u = v(x(t))$, satisfies the inequality*

$$\dot{V} = x'[(A + \Delta A(t))'P + P(A + \Delta A(t))]x + 2x'PBv(x) \leq -\beta \|x\|^2 \quad (1)$$

for all pairs (x, t).

Let us begin by considering the linear plant described by

$$\dot{x} = (A + \Delta A(t))x + Bu, \quad (2)$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control input, and $y \in \mathbf{R}^p$ is the output. $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$ and denote the nominal system matrix, input connection matrix, and output matrix, respectively. We denote $\Delta A(t) \in \mathbf{R}^{n\times n}$ the bounded unknown matrix, describing the unmodeled dynamics of the system. We assume it satisfies the following properties

$$\|\Delta A(t)\| \leq \rho, \quad \forall t. \quad (3)$$

In general, the form of the uncertain matrix $\Delta A(t)$ is unknown. For this reason, we propose an LMI-based strategy to calculate the stabilizing controller gain. The controller design will be based only on the knowledge of the upper bound of the uncertainty matrix. This method is very interesting when the form of uncertainties is not known, but in the meantime, the upper bound of the uncertain parameters should not be very high because the trade-off between defeating unknown parameters by a high-gain controller and having a nice transient behavior is extremely difficult in such situations.

Theorem 1. *System (2) satisfying assumption (3) is globally asymptotically stabilizable by a linear control $u = -B'Px$ provided that there exists a matrix $P > 0$, such that for any controllable pair (A, B), the optimization problem* *maximize* ρ

$$\text{subject to} \begin{bmatrix} P^{-1}A' + AP^{-1} - 2BB' & P^{-1} & I_{n\times n} \\ P^{-1} & -I_{n\times n} & 0_{n\times n} \\ I_{n\times n} & 0_{n\times n} & -\frac{1}{\rho^2}I_{n\times n} \end{bmatrix} < 0 \tag{4}$$

is feasible.

Proof. Let $V = x'Px$ be the Lyapunov function associated to the closed-loop system

$$\dot{x} = (A - BB'P + \Delta A(t))x. \tag{5}$$

Then

$$\begin{aligned} \dot{V} &= x'(A - BB'P + A(t))'Px + x'P(A - BB'P + \Delta A(t))x \\ &\leq x'(A'P + PA - 2PBB'P)x + 2\left\|x'\Delta A(t)Px\right\|. \end{aligned} \tag{6}$$

We have

$$2\left\|x'\Delta A\,(t)\,Px\right\| \leq 2\,\|x\|\,\|\Delta A\,(t)\,Px\| \leq 2\rho\,\|x\|\,\|Px\| \leq \|x\|^2 + \rho^2\,\|Px\|^2. \tag{7}$$

This gives

$$\dot{V} \leq x'(A'P + PA - 2PBB'P + p^2PP + I_{n\times n})x. \tag{8}$$

The system is stable if and only if

$$P^{-1}(A'P + PA - 2PBB'P + \rho^2PP + I_{n\times n})P^{-1} < 0 \tag{9}$$

or

$$P^{-1}A' + AP^{-1} - 2BB' + P^{-1}P^{-1} + \rho^2 I_{n\times n} < 0. \tag{10}$$

Using the Schur complement lemma, inequality (10) is equivalent to equation (4).

In the next sections we shall exploit the structures of uncertainties and the special forms of the nominal matrices to build robust linear controllers using the knowledge of the uncertainty bound.

3. CASE 1: UNCERTAINTIES IN LOWER-TRIANGULAR FORM

In this subsection, we suppose that uncertainties $\Delta A(t)$ have a well-known structured form. Generally, the dynamics of motion is given in terms of uncertain input–output differential equations, which can be easily rewritten in state space form where the nominal matrices A and B are given in controllable canonical form and the matrix of uncertainties appear as a lower-triangular matrix. Here, we consider the case of a single-input multi-outputs system where

$$\Delta A(t) = \begin{bmatrix} \delta a_{11}(t) & 0 & 0 & \dots & 0 \\ \delta a_{21}(t) & \delta a_{22}(t) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ \delta a_{n1}(t) & \delta a_{n2}(t) & \dots & \dots & \delta a_{nn}(t) \end{bmatrix} \tag{11}$$

and

$$\|\Delta A(t)\| \leq \rho \tag{12}$$

We assume that A and B are in the following controllable canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{13}$$

Note that the stabilization of such system can be achieved by several methods, such as backstepping control (Freeman and Kokotovic, 1996; Jankovic et al, 1996), sliding-mode algorithms and Lyapunov techniques (Galimidi and Barmish, 1986; Chen, 1987a, 1987b; Chen and Leitmann, 1987; Xie and Soh, 1994; Najson and Kreindler, 1996; Kogan, 1999). In this subsection, we propose a novel technique to determine the coefficients of the high-gain controller. Consider the linear transformation $z = T(\gamma)x$, where

$$T_{i,j}^{-1}(\gamma) = \begin{cases} (-1)^{i+j} \dfrac{C_{n-i}^{j-i}}{\gamma^{n-j}} & \text{for } n-i > j-i, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

Then its inverse transformation is

$$T_{i,j}^{-1}(\gamma) = \begin{cases} (-1)^{i+j\frac{C_{n-i}^{j-i}}{\gamma^{n-j}}} & \text{for } n-i > j-i, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where C_n^k is the binomial coefficient. In the z-coordinates, system (2) takes the following form

$$\begin{aligned} \dot{z} &= (T(\gamma)AT^{-1}(\gamma) + T(\gamma)\Delta A(t)T^{-1}(\gamma))z + T(\gamma)Bu \\ &= (\mathcal{A}(\gamma) + T(\gamma)\Delta A(t)T^{-1}(\gamma))z + \mathcal{B}u, \end{aligned} \tag{16}$$

where

$$\mathcal{A}(\gamma) = \begin{bmatrix} 0 & \gamma & \gamma & \dots & \gamma \\ 0 & 0 & \gamma & \dots & \gamma \\ 0 & 0 & \ddots & \ddots & \gamma \\ \dots & \dots & \dots & \dots & \gamma \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}. \tag{17}$$

Due to the specific structures of the matrices $T(\gamma)$, $T^{-1}(\gamma)$, and $\Delta A(t)$, we can always find a rational function $\lambda(\gamma)$ such that

$$\left\|T(\gamma)\Delta A(t)T^{-1}(\gamma)\right\| \leq \lambda(\gamma). \tag{18}$$

From the explicit formulae of $T(\gamma)$ and $T^{-1}(\gamma)$, it is easy to show that for any dimension n, we have

$$\begin{aligned} (T(\gamma)\Delta A(t)T^{-1}(\gamma))_{i,j} &= \sum_{k=1}^{n}\sum_{m=1}^{n}(-1)^{(k+j)}(\Delta A(t))_{m,k}C_{n-k}^{j-k}C_{n-i}^{m-i}/\gamma^{m-k}, \\ 1 &\leq i,\ j \leq n. \end{aligned} \tag{19}$$

Since for $m < k$, $\Delta A(t) = 0$, then it is clear that for $m - k \geq 0$, the elements of $T(\gamma)\,\Delta A\,(t)\,T^{-1}(\gamma)$ are either constants or rational functions of γ. Using this fact, we can develop a stabilizing linear controller, which depends upon the parameter γ. The design of the high-gain controller is given in the following theorem.

Theorem 2. *Consider system (2). Under the above assumptions, the linear controller*

$$u = -\gamma\,\mathcal{B}'z \tag{20}$$

stabilizes system (2) at the origin for all $\gamma > \lambda^2(\gamma) + 1$.

Proof. The dynamics of the closed-loop system (2) under the feedback $u = -\gamma\,\mathcal{B}'T(\gamma)x$ is

$$\dot{z} = (\mathcal{A}(\gamma) + T(\gamma)\Delta A(t)T^{-1}(\gamma) - \gamma\,\mathcal{B}\mathcal{B}')z. \tag{21}$$

Remark that for all γ,

$$(\mathcal{A}(\gamma) - \gamma\,\mathcal{B}\mathcal{B}')' + (\mathcal{A}(\gamma) - \gamma\,\mathcal{B}\mathcal{B}') = -\gamma\,\mathcal{B}\mathcal{B}' - \gamma\,I. \tag{22}$$

Then $P = I$ is the solution of the Lyapunov equation

$$P(\mathcal{A}(\gamma) - \gamma\,\mathcal{B}\mathcal{B}')' + (\mathcal{A}(\gamma) - \gamma\,\mathcal{B}\mathcal{B}')P = -\gamma\,\mathcal{B}\mathcal{B}' - \gamma\,I. \tag{23}$$

By adding the matrix I to both sides of equation (23), and considering always that $P = I$ we have

$$\mathcal{A}(\gamma)P + P\mathcal{A}'(\gamma) - \gamma\mathcal{B}\mathcal{B}'P - \gamma P\mathcal{B}\mathcal{B}' + I = -\gamma P\mathcal{B}\mathcal{B}' - \gamma I + I. \tag{24}$$

By replacing I by $P2$ in the left-hand side of equation (24) we can write the matrix $P = I$ as the solution of the algebraic Riccati equation

$$\mathcal{A}(\gamma)P + P\mathcal{A}'(\gamma) - \gamma\mathcal{B}\mathcal{B}'P - \gamma P\mathcal{B}\mathcal{B}' + I = -\gamma P\mathcal{B}\mathcal{B}' - \gamma I + I \tag{25}$$

where $Q(\gamma)$ is a diagonal positive definite matrix

$$Q(\gamma) = \begin{bmatrix} \gamma - 1 & 0 & \dots & 0 \\ 0 & \gamma - 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \gamma - 1 \end{bmatrix}. \tag{26}$$

Since $P = I$ is the solution of the Riccati equation (25), then we rewrite the dynamics of the closed-loop system (21) as

$$\begin{cases} \dot{z} = (\mathcal{A}(\gamma) + T(\gamma)\Delta A(t)T^{-1}(\gamma) - \gamma\mathcal{B}\mathcal{B}'P^{-1})z, \\ \mathcal{A}(\gamma)P + P\mathcal{A}'(\gamma) - P(\gamma\mathcal{B}\mathcal{B}' - I)P + Q(\gamma) = 0. \end{cases} \tag{27}$$

Consider $V = z'P^{-1}z$ as a Lyapunov function for system (27). Then

$$\begin{aligned} \dot{V} &= \dot{z}'P^{-1}z + z'P^{-1}\dot{z} \\ &= z'\left[\mathcal{A}'(\gamma) - \gamma P^{-1}\mathcal{B}\mathcal{B}' + (T(\gamma)\Delta A(t)T^{-1}(\gamma))'\right]P^{-1}z \\ &+ z'P^{-1}\left[\mathcal{A}(\gamma) - \gamma\mathcal{B}\mathcal{B}'P^{-1} + T(\gamma)\Delta A(t)T^{-1}(\gamma)\right]z. \end{aligned} \tag{28}$$

Substituting the second equation of (27) into equation (28), we obtain

$$\begin{aligned} \dot{V} &= z'(-P^{-1}Q(\gamma)P^{-1} - \gamma\mathcal{B}\mathcal{B}' - I)z \\ &+ z'\left[P^{-1}T(\gamma)\Delta A(t)T^{-1}(\gamma) + (T(\gamma)\Delta A(t)T^{-1}(\gamma))'P^{-1}\right]z. \end{aligned} \tag{29}$$

This gives

$$\begin{aligned} \dot{V} &\leq z'(P^{-1}Q(\gamma)P^{-1})z - z'z + 2\|z\|\left\|T(\gamma)\Delta A(t)T^{-1}(\gamma)\right\|\left\|P^{-1}(z)\right\| \\ &\leq -z'(P^{-1}Q(\gamma)P^{-1})z - z'z + 2\lambda\|z\|\left\|P^{-1}z\right\|. \end{aligned} \tag{30}$$

Using the fact that for any $a > 0$ and $b > 0$, we have

$$2ab \leq a^2 + b^2, \tag{31}$$

then by replacing $a = \|z\|$, and $b = \lambda(\gamma)\left\|P^{-1}z\right\|$, we write

$$2\lambda \|z\| \left\|P^{-1}z\right\| \leq \lambda^2(\gamma)z'P^{-1}P^{-1}z + z'z. \tag{32}$$

Finally, we obtain

$$\dot{V} \leq -z'P^{-1}(Q(\gamma) - \lambda^2(\gamma)I)P^{-1}z \tag{33}$$

i.e. for γ sufficiently large such that $\gamma > \lambda^2(\gamma) + 1$, the matrix $Q(\gamma) - \lambda^2(\gamma)I$ is positive definite and, consequently, $\dot{V}$ is always negative, which is the claim.

4. CASE 2: B IN MODAL FORM

In some cases the structures of the nominal matrices simplify the stabilization problem on hand and make the conditions of existence of a stabilizing controller, generally known as the matching conditions, less restrictive. Consider now system (2) with the following assumptions

$$\begin{aligned} \Delta A(t) &= DF(t)E, \quad \forall t \geq 0 \\ F'(t)DD'F(t) &\leq I, \\ E'E &\leq \sqrt{\gamma} I, \end{aligned} \tag{34}$$

for some positive parameter γ. The matrices D and E are real matrices with appropriate dimensions and $F(t)$ is the uncertainty matrix. We assume that there exists a positive definite matrix $R \in R^{m \times m}$ and $\bar{\alpha} < \gamma/2$ such that the condition

$$I/\sqrt{\gamma} \leq BRB' \leq \bar{\alpha}I \tag{35}$$

holds. We suppose that (A, B) is a controllable pair, and the matrix A enjoys the following property

$$\mu(-A) \leqslant \gamma/2, \tag{36}$$

and

$$-A' - I\,(\gamma/2) \text{ is Hurwitz.} \tag{37}$$

Remark 1. Conditions (34) are not restrictive if we choose γ sufficiently large. Remark that condition (35) can be satisfied if B is given in modal form. When B is not given in modal form, one could look for to a certain linear transformation which satisfies condition (35).

Theorem 3. *Under the above assumptions (34), (35), (36), and (53) the linear feedback*

$$u = -RB'P^{-1}x \tag{38}$$

makes the origin $x = 0$ *a stable equilibrium point of system (2) where* $P > 0$ *is the solution of the Lyapunov matrix equation*

$$-\gamma P - AP - PA' + BRB' = 0. \tag{39}$$

Proof. Let us assign $V = x'P^{-1}x$ as a Lyapunov function to the controlled system

$$\dot{x} = (A + \Delta A(t) - BRB'P^{-1})x. \tag{40}$$

Then the time derivative of V is

$$\begin{aligned}\dot{V} &= \dot{x}'P^{-1}x + x'P^{-1}\dot{x}\\ &= x'(A' + \Delta A'(t) - P^{-1}BRB')P^{-1}x\\ &+ x'P^{-1}(A + \Delta A(t) - BRB'P^{-1})x\\ &= x'(-\gamma P^{-1} - P^{-1}BRB'P^{-1} + \Delta A'(t)P^{-1} + P^{-1}\Delta A(t))x.\end{aligned} \tag{41}$$

Using the fact that for given matrices X and Y, we have

$$X'Y + Y'X \leq aX'X + \frac{1}{a}Y'Y, \tag{42}$$

for any $a < 0$. Then, if we put $a = \sqrt{\gamma}$, we obtain the following inequality

$$\Delta A'(t)P^{-1} + P^{-1}\Delta A(t) \leqslant \sqrt{\gamma}\Delta A'(t)\Delta A(t) + \left(1/\sqrt{\gamma}\right) P^{-1}P^{-1}. \tag{43}$$

Taking into account the set of conditions (34), we obtain

$$\begin{aligned}\sqrt{\gamma}\Delta A'(t)\Delta A(t) + \frac{1}{\sqrt{\gamma}}P^{-1}P^{-1} &= \sqrt{\gamma}E'F'(t)D'DF(t)E + \frac{1}{\sqrt{\gamma}}P^{-1}P^{-1}\\ &\leq \gamma I + \frac{1}{\sqrt{\gamma}}P^{-1}P^{-1}.\end{aligned} \tag{44}$$

Using equations (41), this gives

$$\dot{V} = x' \left[-\gamma P^{-1} - P^{-1}\left(BRB' - I/\sqrt{\gamma}\right)P^{-1} + \gamma I\right] x. \tag{45}$$

Using equations (34), the matrix $BRB' - I/\sqrt{\gamma} > 0$. Furthermore, the solution of the algebraic matrix equation (39) can be rewritten as

$$P = \int_0^\infty e^{-\gamma t} e^{-At} BRB' e^{-A't} d\tau \le \left\|BRB'\right\| \int_0^\infty e^{-\gamma t} e^{-At} e^{-A't} d\tau. \tag{46}$$

Since $e^{-At}e^{-A't} > 0$, then we write

$$e^{-At}e^{-A't} \le \lambda_{max}(e^{-At}e^{-A't})I \le e^{2\mu(-A')t} I \tag{47}$$

where $\mu(A) = \lambda_{\max}((A + A')/2)$. We conclude that

$$P \le \left\|BRB'\right\| \int_0^\infty e^{-(\gamma - 2\mu(-A'))t} dt. \quad I = \frac{\|BRB'\|}{\gamma - 2\mu(-A')} I. \tag{48}$$

Using inequality (35), then we have $\|BRB'\| \le \bar{\alpha}$ with $\bar{\alpha} \le \gamma/2$. Finally, P is bounded as follows

$$P \le 2\bar{\alpha} I/\gamma \le I. \tag{49}$$

Consequently, $P^{-1} > I$ and $-\gamma P^{-1} + \gamma I < 0$. This immediately gives

$$\dot{V} \le 0. \tag{50}$$

Remark 2. Remark that P always exists and is positive definite whatever $\gamma > 0$. The constant γ is a measure of the boundedness of A, B and $\Delta A(t)$ and can be easily obtained by a simple calculation of their respective norms.

When uncertainties do not verify conditions (36) and (35), the existence of a robust linear controller depends upon the solvability of an LMI problem. This result is given in the following theorem.

Theorem 4. *Under condition (34), system (2) is quadratically stabilizable by the linear controller*

$$u = -RB'P^{-1}x,$$

if there exists a constant $\beta > 0$ *and two positive definite matrices* $R > 0$, *and* $P > 0$, *such that*

$$-\gamma P^{-1} - P^{-1}\left(BRB' - I/\beta\right)P^{-1} + \beta\sqrt{\gamma}I \leq 0, \tag{51}$$

and

$$-\gamma P - AP - PA' + BRB' = 0. \tag{52}$$

The proof of Theorem 4 is quite similar to the proof of Theorem 3 except that we would replace $\sqrt{\gamma}$ in equation (43) by β. Choosing β small increases the chance of finding a positive definite matrix P that verifies equation (51).

When uncertainties are present in both the state matrix A and the control matrix B, the problem of finding a robust stabilizing controller turns on a problem (2) by adding just an integrator to system (2). Augmenting the state vector by an integrator permits us to remove uncertainty from the control matrix B and to simplify the controller design as mentioned in Theorem 3. This result is detailed in the following corollary.

Collary 1. Let $\mathcal{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $\Delta\mathcal{A} = \begin{bmatrix} \Delta A(t) & \Delta B(t) \\ 0 & 0 \end{bmatrix}$. If the following assumptions hold:

(i) pair $(\mathcal{A}, \mathcal{B})$ is controllable;

(ii) there exist real matrices $\mathcal{D}$ and $\mathcal{E}$ with appropriate dimensions such that

$$\begin{aligned} \Delta\mathcal{A}(t) &= \mathcal{D}\mathcal{F}(t)\mathcal{E}, \quad \forall t \geq 0; \\ \mathcal{F}'(t)\mathcal{D}'\mathcal{D}\mathcal{F}(t) &\leq I; \\ \mathcal{E}'\mathcal{E} &\leq \sqrt{\gamma}I; \end{aligned} \tag{53}$$

(iii) there exists a positive definite matrix $\mathcal{R} \in R^{m+1\times m+1}$ and $\bar{\alpha} < \frac{\gamma}{2}$ such that

$$\frac{1}{\sqrt{\gamma}}I \leq BRB' \leq \bar{\alpha}I;$$

(iv) there exists $\gamma > 0$ such that

$$\mu(-\mathcal{A}) \leq \frac{\gamma}{2} \text{ and } -\mathcal{A}' - \frac{\gamma}{2}I \text{ is Hurwitz;} \tag{54}$$

then the uncertain linear system

$$\dot{x} = (A + \Delta A(t))x + (B + \Delta B(t))u \tag{55}$$

is quadratically stabilizable by the controller

$$u = -\mathcal{R}\mathcal{B}'\mathcal{P}^{-1}x, \tag{56}$$

where $\mathcal{P}$ is the solution of the matrix equation

$$-\gamma\mathcal{P} - \mathcal{A}\mathcal{P} - \mathcal{P}\mathcal{A}' + \mathcal{B}\mathcal{R}\mathcal{B}' = 0. \tag{57}$$

Proof. Consider the system

$$\begin{aligned} \dot{x} &= (A + \Delta A(t))x + (B + \Delta B(t))u, \\ \dot{u} &= v. \end{aligned} \tag{58}$$

If we arrange X under the following form

$$X = \begin{bmatrix} x \\ u \end{bmatrix} \tag{59}$$

then we obtain

$$\dot{X} = (\mathcal{A} + \Delta\mathcal{A}(t))X + \mathcal{B}v, \tag{60}$$

Consequently, the design of a stabilizing controller v is exactly the same as developed for system (2).

5. STABILIZATION OF UNCERTAIN THRUST VECTORING AIRCRAFT

Consider the longitudinal model of the L1011 thrust vectoring aircraft described by the linear model (Jonckheere and Yu, 1999)

$$\begin{bmatrix} \dot{\delta}_\theta \\ \dot{q} \\ \dot{\delta}_\alpha \\ \dot{\delta}_v \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.3385 & 6.374 & -1.37 \\ 0 & 1 & -1.193 & -0.06639 \\ -0.0332 & 0 & 0.03766 & -0.001726 \end{bmatrix} \begin{bmatrix} \delta_\theta \\ q \\ \delta_\alpha \\ \delta_v \end{bmatrix} + \Delta A(t) \right) + \begin{bmatrix} 0 & 0 \\ -10.29 & -3.286 \\ -0.07535 & -0.03393 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{61}$$

where the four state variables are the incremental pitch δ_θ, the pitch rate q, the incremental angle of attack δ_α, and the incremental velocity δ_v. The input variables are the elevator deflection w_1 and the thrust vectoring angle w_2. We assume that the state matrix $A + \Delta A(t)$ is uncertain with a well-known bound $\|\Delta A(t)\| \leq 0.2 = \rho$. The open-loop eigenvalues of A are

$$
\begin{array}{lcll}
\lambda_1 & = & 1.798 & \text{(unstable short period)} \\
\lambda_2 & = & -3.3274 & \text{(stable short period)} \\
\lambda_{3,4} & = & -0.00174 \pm 0.1068i & \text{(phugoid).}
\end{array}
$$

The example given above is a typical example of unstable aircraft. We will apply the result of Theorem 1 to deliver the explicit expression of the controller that stabilizes system (61). Using the package of LMI design with MATLAB written by (P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali), then the solution of the optimization problem (4) is

$$
P^{-1} = \begin{bmatrix} 0.5428 & -0.9513 & 0.2579 & 0.0385 \\ -0.9513 & 9.9063 & -1.2199 & 0.0108 \\ 0.2579 & -1.2199 & 1.6766 & -0.0408 \\ 0.0385 & 0.0108 & -0.0408 & 0.0242 \end{bmatrix}. \tag{62}
$$

For this solution, the eigenvalues of the matrix

$$
\begin{bmatrix} P^{-1}A' + AP^{-1} - 2BB' & P^{-1} & I_{4\times 4} \\ P^{-1} & -I_{4\times 4} & 0_{4\times 4} \\ I_{4\times 4} & 0_{4\times 4} & -\frac{1}{\rho^2} I_{4\times 4} \end{bmatrix} \tag{63}
$$

are all negative reals and the poles of the closed-loop system are

$$
\begin{aligned}
\lambda_1 &= -13.52887488938904 \\
\lambda_2 &= -0.05261256365784 \\
\lambda_3 &= -2.06467331840704 \\
\lambda_4 &= -1.06832602558051.
\end{aligned} \tag{64}
$$

6. CONCLUSION

Uncertain aircraft systems consisting of a continuous-time plant controlled by robust linear controllers have been extensively studied in the literature. This class of automatic control systems has been called upon to address issues of reliability for aircraft control systems in the face of random disturbances and other abrupt changes in the system parameters with flight conditions. In this paper, we have studied the problem of the stabilization of uncertain systems having both unknown uncertainty structures and well-defined structural uncertainties. The whole design was based on the LMI design and the Lyapunov approach.

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